MATH 20D Spring 2023 Lecture 12.

A closer look at homogeneous equations and the Wronskian

Outline

Contents

- A second order linear differential equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t) \quad (1)$$

is **homogeneous** if $g(t) \equiv 0$. Otherwise we say (1) is **inhomogeneous**.

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Theorem

Suppose $y_p(t)$ is a particular solution to (1) and let $y_h(t)$ denote a general solution to the homogeneous equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0. \quad (2)$$

Then $y(t) = y_p(t) + y_h(t)$ is a general solution to (1).

Last Time

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$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

where $y_1(t)$ and $y_2(t)$ are **linearly independent** solutions to (2).

- Suppose $y_1(t)$ and $y_2(t)$ are two solutions to the ODE

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \tag{3}$$

where $b(t)/a(t)$ and $c(t)/a(t)$ are continuous on $\mathbb{R}$.

Lemma

If C_1 and C_2 are constants then $y(t) = C_1y_1(t) + C_2y_2(t)$ is a solution to (3).

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Proof. Let $y(t) = C_1y_1(t) + C_2y_2(t)$ so that

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Therefore

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A Closer Look at Homogeneous Equations

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$$\begin{aligned} &a(t)y''(t) + b(t)y'(t) + c(t)y(t) \\ &= C_1(a(t)y_1''(t) + b(t)y_1'(t) + c(t)y_1(t)) + C_2(a(t)y_2''(t) + b(t)y_2'(t) + c(t)y_2(t)) \end{aligned}$$

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So $y(t) = C_1y_1(t) + C_2y_2(t)$ is a solution to (3).

Question

Suppose $y_1(t)$ *and* $y_2(t)$ *are solutions to the homogeneous equation*

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- The expression $C_1y_1(t) + C_2y_2(t)$ always defines a solution to (4), so since solutions to IVPs are unique, it suffices to find C_1 and C_2 satisfying

$$\begin{cases} C_1y_1(0) + C_2y_2(0) = Y_0 \\ C_1y_1'(0) + C_2y_2'(0) = Y_1. \end{cases}$$

In summary, if $y_1(t)$ and $y_2(t)$ are two solutions to the ODE

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \tag{5}$$

then the expression $C_1 y_1(t) + C_2 y_2(t)$ defines a general solution to (5) provided

$$\begin{cases} C_1 y_1(0) + C_2 y_2(0) = Y_0 \\ C_1 y_1'(0) + C_2 y_2'(0) = Y_1. \end{cases} \tag{6}$$

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$$C_1 = \frac{Y_0y_2'(0) - Y_1y_2(0)}{y_1(0)y_2'(0) - y_1'(0)y_2(0)} \quad \text{and} \quad C_2 = \frac{Y_1y_1(0) - Y_0y_1'(0)}{y_1(0)y_2'(0) - y_1'(0)y_2(0)}.$$

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We will see that (7) is satisfied when y_1 and y_2 are linearly independent.

Contents

Definition

- *Let $u_1(t)$ and $u_2(t)$ are differentiable functions defined on an interval I.*
- *The* ***Wronskian*** *of $u_1(t)$ and $u_2(t)$ is the function*

$$W[u_1, u_2]\colon I \to \mathbb{R}, \qquad W[u_1, u_2](t) = u_1(t)u_2'(t) - u_2(t)u_1'(t).$$

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- Show that $u_1(t) = t^2$ and $u_2(t) = t^3$ are linearly indepedent on $\mathbb{R}$.
- Calculate

$$W[t^2, t|t|](t)$$

are the functions $u_1(t) = t^2$ and $u_2(t) = t|t|$ linearly dependent on $\mathbb{R}$.

Theorem

Suppose $u_1, u_2 \colon I \to \mathbb{R}$ *are differentiable functions defined on an interval* I *and assume* u_1 *and* u_2 *occur as solutions to an ODE of the form*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{8}$$

with $p(t)$ *and* $q(t)$ *continuous. Then either*

$$W[u_1, u_2](t) \equiv 0 \qquad \text{or} \qquad W[u_1, u_2](t) \neq 0 \qquad \text{for all } t \in \mathbb{R}$$

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Example

True or False: $u_1(t) = t^2$ and $u_2(t) = t^3$ are solutions to an equation of the form

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with $p(t)$ and $q(t)$ continuous on $\mathbb{R}$.

Earlier in lecture we saw that if $y_1(t)$ and $y_2(t)$ are two solutions to the ODE

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = 0 \tag{9}$$

with $p(t) = b(t)/a(t)$ and $q(t) = c(t)/a(t)$ continuous on $\mathbb{R}$, then the expression $C_1y_1(t) + C_2y_2(t)$ defines a general solution to (9) provided

$$\begin{cases} C_1y_1(0) + C_2y_2(0) = Y_0 \\ C_1y_1'(0) + C_2y_2'(0) = Y_1. \end{cases} \tag{10}$$

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- We recongnize the identity

$$W[y_1, y_2](0) = y_1(0)y_2'(0) - y_1'(0)y_2(0). \tag{11}$$

Earlier in lecture we saw that if $y_1(t)$ and $y_2(t)$ are two solutions to the ODE

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Hence if y_1 and y_2 are linearly independent then we can apply the theory on the previous slide to conclude that $W[y_1, y_2](0) \neq 0$. So C_1 and C_2 will be well defined.

Example

Let $\alpha \in \mathbb{R}$ and $\beta > 0$ be constants. Calculate the Wronskian

$$W[e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)](t).$$